

The effect of universe inhomogeneities on cosmological distance measurements

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Using the focusing equation, the equation for the cosmological angular diameter distance² is derived, based on the ideas of Academician Zel'dovich, namely, that the distribution of matter at small angles is not homogeneous, and the light cone is close to being empty. We propose some ways of testing a method for measuring the angular diameter distances and show that the proposed method leads to results that agree better with the experimental data than those obtained by the usual methods.

The abundance of observational data in modern cosmology allows for testing a number of ideas put forward in the past. One of such ideas is the approach of Academician Ya. B. Zel'dovich to measurements of the cosmological angular diameter distances [1], which takes into account that null geodesics propagate in a homogeneous Friedmann universe, but the null geodesic congruence (or light cone) from the source experiences a smaller focusing than in a homogeneous universe.³ Such an effect is possible if the density of matter inside a light cone is smaller than the mean density in the Friedmann universe.

Ref. [2] suggested a derivation of a generalized differential equation using such tools as the null geodesics and the ratio of longitudinal and transverse angular momentum of a photon. We will show that this equation can also be obtained on the basis of the focusing equation [6], which follows from

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²Not to be confused with the “angular distance” defined as a distance between points on the celestial sphere and measured in radians or degrees.

³A light cone is understood here and henceforth as “a cone of light rays” or a cone that bounds a beam of null geodesics, not to be confused with the light (or null) cone of relativity theory, which separates spacelike and timelike directions.

the Sachs equations [3] (a special case of the Raychaudhuri equations [5]):

$$\frac{d^2}{d\lambda^2}\sqrt{S} = -\left(|\sigma|^2 + \frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta\right)\sqrt{S}, \quad (1)$$

where S is the light cone cross-section area, λ is the affine parameter, k^α is the null wave vector, Greek indices run over the values $(0, 1, 2, 3)$, and σ is the shear defined as follows:

$$|\sigma|^2 = \frac{1}{2}k_{\alpha;\beta}k^{\alpha;\beta} - \frac{1}{4}(k_{;\alpha}^\alpha)^2. \quad (2)$$

In the Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - a(t)^2[dr^2 + f^2(r)(d\theta^2 + \sin^2\theta d\phi^2)] \quad (3)$$

the wave vector has the following components for the arriving geodesics: $k_{in}^\alpha = (-1/a, 1/a^2, 0, 0)$; the affine parameter is related to time through the scale factor: $d\lambda = -a dt$. Directly calculating the covariant derivatives in the metric (3), we verify that $|\sigma|^2 = 0$:

$$|\sigma|^2 = \frac{(f\dot{a} - f')^2}{f^2a^4} - \frac{(f\dot{a} - f')^2}{f^2a^4} = 0. \quad (4)$$

Expressing the light cone cross-section in terms of the linear size of the source $S = \frac{\pi l^2}{4}$ (see [2] for details) and substituting (4) into (1), we obtain

$$\ddot{l} - \frac{\dot{a}}{a}\dot{l} + a^2\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta l = 0. \quad (5)$$

Using in (5) the definition of an angular diameter distance $d_a = l/\phi$, we arrive at

$$\ddot{d}_a - \frac{\dot{a}}{a}\dot{d}_a + a^2\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta d_a = 0. \quad (6)$$

Contracting the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa T_{\alpha\beta} \quad (7)$$

with $k^\alpha k^\beta$ (using the null nature of the wave vector, $g_{\alpha\beta}k^\alpha k^\beta = 0$), we obtain

$$\frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta = \frac{\kappa}{2}T_{\alpha\beta}k^\alpha k^\beta. \quad (8)$$

Here κ is the Einstein gravitational constant. It should be noted that the beam focusing is affected by only the local value of the Ricci tensor, or, due to the Einstein equations, by the local value of the energy-momentum tensor. The result (8) allows us to convert the equation for the cosmological angular diameter distance (6) to the form

$$\ddot{d}_a - \frac{\dot{a}}{a}\dot{d}_a + 4\pi G a^2 T_{\alpha\beta} k^\alpha k^\beta d_a = 0, \quad (9)$$

where $T_{\alpha\beta}$ is the local value of the energy-momentum tensor inside the light cone. One can introduce the parameter α showing how much matter is there inside the cone:

$$\alpha = \frac{T_{\alpha\beta}k^\alpha k^\beta}{T_{\alpha\beta}^{\text{full}}k^\alpha k^\beta}. \quad (10)$$

Using the definition of the energy-momentum tensor for a perfect fluid, $T_\alpha^\beta = \text{diag}(\rho, -p, -p, -p)$, we obtain:

$$a^2 T_{\alpha\beta} k^\alpha k^\beta = p + \rho. \quad (11)$$

Making explicit the components for the Λ CDM model, we get:

$$p + \rho = p_\Lambda + p_M + p_R + \rho_\Lambda + \rho_M + \rho_R. \quad (12)$$

Using the equations of state for baryonic matter ($p_M = 0$), dark energy ($p_\Lambda = -\rho_\Lambda$), and radiation ($p_R = \rho_R/3$), we convert the relation (12) to

$$p + \rho = \frac{4}{3}\rho_R + \rho_M, \quad (13)$$

where

$$\begin{aligned} \rho_M &= \frac{3H_0^2\Omega_M}{8\pi G} \left(\frac{a_0}{a}\right)^3, \\ \rho_R &= \frac{3H_0^2\Omega_R}{8\pi G} \left(\frac{a_0}{a}\right)^4. \end{aligned} \quad (14)$$

Using (11) and (13), Eq. (9) acquires the form

$$\ddot{d}_a - \frac{\dot{a}}{a}\dot{d}_a + 4\pi G \left(\frac{4}{3}\rho_R + \rho_M\right) d_a = 0 \quad (15)$$

Thus it has been established that dark energy does not participate in focusing of the light rays (which makes clear the question raised in [11]). Since inside the light cone, as a rule, ρ_M and ρ_R tend to zero, the value of d_a will be larger than in Friedmann's homogeneous model [2].

Let us now discuss a number of tests for approaches to calculations of the angular diameter distance which follow from the data on the Sunyaev-Zel'dovich effect (SZE) for galaxy clusters [12–14]. The angular diameter distance may be expressed through the SZE data [12]:

$$\begin{aligned} d_a^{\text{SZE}} &= \frac{(\Delta T_0)^2}{S_{X0}} \left(\frac{m_e c^2}{k_B T_{e0}} \right)^2 \\ &\times \frac{\lambda_{eH0} \mu_e / \mu_H}{4\pi^{3/2} f_{(x, T_e)}^2 T_{\text{CMB}}^2 \sigma_T^2 (1+z)^4} \frac{1}{\theta_c} x \\ &\times \left[\frac{\Gamma(3\beta/2)}{\Gamma(3\beta/2 - 1/2)} \right]^2 \frac{\Gamma(3\beta - 1/2)}{\Gamma(3\beta)}, \end{aligned} \quad (16)$$

where $\Gamma(x)$ is the gamma function, S_{X0} is the X-ray surface brightness of the cluster center, z is the redshift, λ_{eH} is the cooling function of the cluster center, σ_T is the total scattering cross-section, k_B is the Boltzmann constant, ΔT_0 is the SZE temperature difference, θ_c is the angular size of the galactic nucleus, m_e is the electron mass, $f(x, T_e)$ is the SZE frequency dependence, and T_{CMB} is the temperature of the microwave background radiation.

Thus there emerges a test for the angular diameter distance connected with the Hubble constant H_0 . Let us write down the solution of (15) for an empty light cone in Λ CDM [2]:

$$d_a^{\text{empty}} = \frac{1}{H_0} \int_{\frac{1}{1+z}}^1 \frac{dx}{\sqrt{\Omega_S}} \quad (17)$$

where $\Omega_S = \Omega_\Lambda + \Omega_k x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}$, while for a light cone filled with matter whose density is equal to the mean density of the Universe, the

solution of (15) has the form

$$\begin{aligned}
 d_a^{\text{full}} &= \frac{1}{1+z} \frac{1}{H_0 \sqrt{\Omega_k}} \sin \int_{\frac{1}{1+z}}^1 \sqrt{\Omega_k} \frac{dx}{x^2 \sqrt{\Omega_S}} \\
 &\quad \text{for } k = 1, \\
 d_a^{\text{full}} &= \frac{1}{1+z} \int_{\frac{1}{1+z}}^1 \frac{dx}{H_0 x^2 \sqrt{\Omega_S}} \quad \text{for } k = 0, \\
 d_a^{\text{full}} &= \frac{1}{1+z} \frac{1}{H_0 \sqrt{\Omega_k}} \sinh \int_{\frac{1}{1+z}}^1 \sqrt{\Omega_k} \frac{dx}{x^2 \sqrt{\Omega_S}}, \\
 &\quad \text{for } k = -1.
 \end{aligned} \tag{18}$$

This allows us to compare the values of the Hubble constant predicted by the standard solution (18) and the new formula (17). The value of the cosmological angular diameter distance is calculated directly from the SZE, which means that equating d_a^{empty} and d_a^{full} to d_a^{ZSE} , we can find H_0 . Therefore, for an empty light cone we obtain

$$H_0^{\text{empty}} = \left(\int_{\frac{1}{1+z}}^1 \frac{dx}{\sqrt{\Omega_S}} \right) / d_a^{\text{ZSE}}, \tag{19}$$

while for a full light cone

$$\begin{aligned}
 H_0^{\text{ZSE}} &= \frac{1}{(1+z)d_a^{\text{ZSE}}} \frac{1}{\sqrt{\Omega_k}} \sin \int_{\frac{1}{1+z}}^1 \sqrt{\Omega_k} \frac{dx}{x^2 \sqrt{\Omega_S}} \\
 &\quad \text{for } k = 1, \\
 H_0^{\text{ZSE}} &= \frac{1}{(1+z)d_a^{\text{ZSE}}} \int_{\frac{1}{1+z}}^1 \frac{dx}{x^2 \sqrt{\Omega_S}} \quad \text{for } k = 0, \\
 H_0^{\text{ZSE}} &= \frac{1}{(1+z)d_a^{\text{ZSE}}} \frac{1}{\sqrt{\Omega_k}} \sinh \int_{\frac{1}{1+z}}^1 \sqrt{\Omega_k} \frac{dx}{x^2 \sqrt{\Omega_S}} \\
 &\quad \text{for } k = -1.
 \end{aligned} \tag{20}$$

The calculation of the simple averages from the data for clusters of galaxies [12] allows us to conclude that more consistent values of the Hubble constant are given by Eqs.(19) than (20). A detailed analysis of the galactic cluster data in the context of using Eqs.(19) can serve as a material for further experimental studies.

The next test is connected with the duality between the cosmological angular diameter distance d_a and the luminosity distance d_l :

$$\eta = \frac{d_l}{d_a} (1+z)^{-2} = 1, \tag{21}$$

which follows from the Eddington identity [4]:

$$r_s^2 = r_o^2 (1+z)^2, \tag{22}$$

where r_s is the distance to the source and r_o is the distance to the observer, which is determined through the solid angle and the cross-section area, $dS = r^2 d\Omega$ (for more details see [7]).

In [10], an attempt is undertaken to test the validity of the identity (21) on the basis of the data from galaxy clusters [12] using the formula

$$\eta(z) = \sqrt{\frac{d_a^{\text{Th}}}{d_a^{\text{data}}}}, \tag{23}$$

where d_a^{Th} is obtained from theoretical calculations according to (17) or (18). An analysis [9] shows that the new method of calculations of the angular diameter distance allows one to experimentally confirm the identity (21) with a greater accuracy than the standard method.

Another approach to verification of the identity has been proposed in [8], using the surface brightness data in the X-ray spectrum together with SZE data [13, 14]. To assess the validity of the identity (21), one uses the mass fraction of gas in the galaxy, $f = M_{\text{gas}}/M_{\text{Tot}}$, and the ratio

$$\eta(z) = \frac{f_{\text{SZE}}}{f_{\text{X-ray}}}, \tag{24}$$

where f_{SZE} is the mass fraction of gas measured with the aid of the SZE, and $f_{\text{X-ray}}$ is the same calculated assuming the validity of (21) [14, 15].

If we insert the correction connected with applying the new method of calculation of the angular diameter distance (17), an analysis shows that the z dependence of $\eta(z)$ becomes closer to unity. This argues in favor of the new method of calculation of the angular diameter distance. The very distribution of values of η is shifted to unity, showing that the duality identity for cosmological distances holds with an accuracy of 1σ , in contrast to the result (2σ) of the original work [8].

In conclusion, we would like to note that, based on the aforementioned reasoning, Zel'dovich's idea receives a confirmation. In contrast to the papers [16, 17], developing the ideas of Dyer and Roeder, we obtain simpler calculation formulas which reflect the physical meaning of measurements of the cosmological angular diameter distance in the Friedmann universe taking into account the inhomogeneities. Our approach makes it possible to pass on to the stage of experimental verification.

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